

Symmetries of Non-Linear Systems: Group Approach to their Quantization

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Abstract

We report briefly on an approach to quantum theory entirely based on symmetry grounds which improves Geometric Quantization in some respects and provides an alternative to the canonical framework. The present scheme, being typically non-perturbative, is primarily intended for non-linear systems, although needless to say that finding the basic symmetry associated with a given (quantum) physical problem is in general a difficult task, which many times nearly emulates the complexity of finding the actual (classical) solutions. Apart from some interesting examples related to the electromagnetic and gravitational particle interactions, where an algebraic version of the equivalence principle naturally arises, we attempt to the quantum description of non-linear sigma models. In particular, we present the actual quantization of the partial-trace non-linear $SU(2)$ sigma model as a representative case of non-linear quantum field theory.

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1 Introduction

In this brief report we attempt to illustrate the features of a group-theoretical approach to the quantum description of fundamental physical systems, which is being developed over the last decades [1]. The underlying motivation for pushing ahead the present Group Approach to Quantization (GAQ) is twofold. On the one hand we face a basic argument of beauty; we attempt to contribute to the big effort that had been devoted to place Quantum Mechanics in a similar geometrical status to that of Classical Mechanics or, even, General Relativity, in that which was known as Geometric Quantization (GQ) [2, 3, 4, 5]. On the other, there exists more practical reasons supported by a great variety of physical examples where the old (canonical) quantization finds serious difficulties in describing them properly. In fact, the limitations of Canonical Quantization were soon stated neatly through the so-called “no-go” theorems [6] (see also [7]).

The main ingredient in GAQ is the group structure taken to the ultimate consequences, that is to say, symmetry is intended to contribute to Physics as a building block rather than a practical

tool for finding additional solutions to partially solved (symmetrical) problems. Even more, this approach attempts to describe a quantum physical system from the group manifold itself and its canonical structures, aiming at reducing the problem of establishing the physical postulates to that of choosing specific groups. In addition, it should be considered as a method for describing directly the quantum dynamics since the intermediate step of solving the classical equation of motion is not required. In fact, the quantum nature of a given system can be associated with the actual (compact) topology of (part of) the addressing symmetry group, whereas the classical limit is obtained by simply taking a local version (in the sense of taking a local chart) of the symmetry (“opening” the multiplicative $U(1)$ central subgroup to the additive real line \mathbb{R}).

From a technical point of view, the present method also represents significant advantages. In particular, the biggest obstruction found by Geometric Quantization in dealing with non-linear systems, that of achieving the complete reduction of the geometric representation (polarization), can now be much better addressed on the grounds of the algebraic group structure. This is mainly due to the existence of two mutually commuting (left- and right-) actions, so that the infinitesimal generator of one of them can be used to construct the Poisson (classical) algebra representation (pre-quantization in the sense of Geometric Quantization), whereas the other can be employed to reduce completely the representation (true quantization).

Although the requirement of the additional structure of Lie group might be seen as a drawback, it should be remarked that after all, the Lie algebra structure is one of the few bricks shared by all quantization methods, which look for unitary and irreducible representations of a given Lie (Poisson) algebra somehow characterizing a physical system.

This paper is organized as follows. In Sec. 2 we motivate the central extensions of classical symmetry groups with the example of the Galilei group as well as the extension of classical phase space with an extra variable. In Sec. 3 the fundamentals of the Group Approach to Quantization scheme are presented. Sec. 4 is devoted to illustrating the way in which GAQ describes physical systems bearing a finite number of degrees of freedom. We start with the example of the free particle and then proceed to introduce interactions through some sort of revisited Minimal Coupling Principle. In particular, the particle moving in an electromagnetic field, as well as the geodesic motion in a gravitational field are analyzed. In the last case, a very simple algebraic version of the Equivalence Principle naturally arises. In Sec. 5 we present examples of infinite-dimensional systems. After studying the example of the Klein-Gordon field we end up with the case the Non-Linear Sigma Model (NLSM) as an example of a genuine non-linear field.

We wish to mention that the present GAQ method has been applied to numerous physical systems that can not be reported here. Among them, we refer the reader to Refs. [8, 9, 10, 11, 12, 13].

2 The role of central extensions of “classical” symmetries

Let us consider the symmetry of the Lagrangian of the free particle in $1 + 1$ dimensions:

$$\mathcal{L} = \frac{1}{2}m \dot{x}^2$$

Under the classical Galilean transformations

$$x' = x + A + Vt, \quad t' = t + B, \quad (1)$$

The Lagrangian moves to $\mathcal{L}' = \frac{1}{2}m(\dot{x} + V)^2 = \mathcal{L} + \frac{d}{dt}(\frac{1}{2}mV^2t + mVx)$. That is, \mathcal{L} is not strictly invariant, but *semi-invariant*, due to the presence of the total derivative.

In infinitesimal terms something similar happens. Taking the Lie derivative of \mathcal{L} with respect to the generators of the group results in:

$$\begin{aligned} X_B &= \frac{\partial}{\partial t} &\Rightarrow X_B \mathcal{L} &= 0 \\ X_A &= \frac{\partial}{\partial x} &\Rightarrow X_A \mathcal{L} &= 0 \\ X_V &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial \dot{x}} &\Rightarrow X_V \mathcal{L} &= \frac{d}{dt}(mx) \neq 0 \end{aligned}$$

The same is also valid for the *Poincaré-Cartan form*:

$$\Theta_{PC} = p dx - H dt = \frac{\partial \mathcal{L}}{\partial \dot{x}} dx - (\dot{x} p - \mathcal{L}) dt = \frac{\partial \mathcal{L}}{\partial \dot{x}} (dx - \dot{x} dt) + \mathcal{L} dt$$

whose Lie derivative is:

$$L_{X_B} \Theta_{PC} = 0, \quad L_{X_A} \Theta_{PC} = 0, \quad L_{X_V} \Theta_{PC} = d(mx) \neq 0. \quad (2)$$

The quantum free particle suffers from the same “pathology” although it manifests in a different manner. Let us apply the Galilean transformations (1) to the Schrödinger equation. We get:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi \quad \rightarrow \quad i\hbar \frac{\partial \Psi}{\partial t'} = -\frac{\hbar^2}{2m} \nabla'^2 \Psi - i\hbar V \frac{\partial \Psi}{\partial x'}.$$

The extra term can be compensated if we also transform the wave function by means of a non-trivial phase:

$$\Psi \rightarrow \Psi' = e^{i\frac{m}{\hbar}(Vx + \frac{1}{2}V^2t)} \Psi. \quad (3)$$

Then, we recover the original (fully primed) Schrödinger equation $i\hbar \frac{\partial \Psi'}{\partial t'} = -\frac{\hbar^2}{2m} \nabla'^2 \Psi'$. Joining together the Galilean transformations (1) and the phase transformation (3) we obtain a group of *strict* symmetry whose group law is:

$$\begin{aligned} B'' &= B' + B \\ A'' &= A' + A + V'B \\ V'' &= V' + V \\ \zeta'' &= \zeta' \zeta e^{i\frac{m}{\hbar}[A'V + B(V'V + \frac{1}{2}V'^2)]}, \end{aligned} \quad (4)$$

where the last line has the general form $\zeta'' = \zeta' \zeta e^{i\frac{m}{\hbar} \xi(g', g)}$, with $\zeta \equiv e^{i\phi} \in U(1)$ and the function ξ being that which is customarily named 2-cocycle on the Galilei group, characterized by the mass m [14, 15, 16]. A constant \hbar with the dimensions of an action has to be introduced to keep the exponent dimensionless.

The infinitesimal version of the group law (4) is expressed by means of the extended Lie algebra commutators:

$$\left[\tilde{X}_B, \tilde{X}_A\right] = 0, \quad \left[\tilde{X}_B, \tilde{X}_V\right] = \tilde{X}_A, \quad \left[\tilde{X}_A, \tilde{X}_V\right] = -m\tilde{X}_\phi. \quad (5)$$

between the extended generators:

$$\begin{aligned} \tilde{X}_B &= \frac{\partial}{\partial t} \\ \tilde{X}_A &= \frac{\partial}{\partial x} \\ \tilde{X}_V &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial \dot{x}} - mxi\zeta \frac{\partial}{\partial \zeta} \\ \tilde{X}_\phi &= i\zeta \frac{\partial}{\partial \zeta}. \end{aligned}$$

One of the relevant points concerning both the *strict invariance* and, consequently, the *centrally extended symmetry* is that the corresponding *extended Lie algebra* now properly represents the Poisson algebra generated by $\langle H \equiv \frac{P^2}{2m}, K \equiv x - \frac{p}{m}t, P \equiv p, 1 \rangle$ when acting as ordinary derivations on complex functions, provided that we impose that the new generator \tilde{X}_ϕ acts on Ψ as $\tilde{X}_\phi \Psi = i\Psi$, or, in finite terms, $\Psi(\zeta g) = \zeta \Psi(g)$. Notice that the unextended algebra, with the commutator $[X_A, X_V] = 0$, is not an isomorphic image of the corresponding Noether invariants H, P, K algebra.

There is yet another remarkable advantage of requiring the strict symmetry of a given arbitrary classical system. In fact, such a symmetry can only be realized faithfully if we extend the classical phase space M parameterized by K, P (or solution manifold) by an extra variable, to be identified with ϕ or $\zeta = e^{i\phi}$. In the compact ($U(1)$) case, that is, the choice of ζ , we thus arrive at the notion of a *quantum manifold* Q [2, 5]. In this manifold, locally parameterized by $K, P, \zeta \equiv e^{i\phi}$, an extended Liouville form (Θ_{PC}) defines the Liouville form ϑ on the solution manifold except for a total differential)

$$\Theta = \vartheta + \frac{d\zeta}{i\zeta} \quad (\text{or } PdK + d\phi)$$

substitutes successfully the ordinary one in the search for an invertible duality between Hamiltonian functions and Hamiltonian vector fields. In fact, the Hamiltonian correspondence

$$f \mapsto X_f \text{ such that } i_{X_f} d\vartheta = -df$$

has the real numbers \mathbb{R} as kernel. However the correspondence

$$f \mapsto \tilde{X}_f \text{ such that } i_{\tilde{X}_f} d\Theta = -df, i_{\tilde{X}_f} \Theta = f$$

has unique solution.

This is, so to speak, the starting point for GQ, where the pair (symplectic manifold) $(M, \omega \equiv d\vartheta)$ is replaced with the pair (Q, Θ) as a $U(1)$ -principal bundle with connection (quantum manifold) under the requirement that the curvature of Θ defines on M the symplectic form ω with integer co-homology class (tantamount to say that the integration of ϑ on closed curves

results in an integer; this is a modern, geometric version of the Bohr-Sommerfeld rules [5]). The association $f \mapsto \hat{f} \approx \tilde{X}_f$ defines the pre-quantum operators, as derivations on complex $U(1)$ -functions on Q , which realize a unitary representation of the Poisson bracket although non-irreducible. The true quantization, that is to say, the irreducibility, is intended to be achieved after the polarization condition is imposed (see Refs. [2, 5] and the analogous condition in next section).

It should be mentioned that the possibility exists of extending the classical phase space by the real line and the classical group by the non-compact additive group \mathbb{R} . In that case the constant \hbar is no longer needed and the resulting theory describes the classical limit in a *global* version of the Hamilton-Jacobi formulation (see Ref. [1]).

3 Group Approach to Quantization

The essential idea underlying a group-theoretical framework for quantization consists in selecting a given subalgebra $\tilde{\mathcal{G}}$ of the classical Poisson algebra including $\langle H, p_i, x^j, 1 \rangle$ and finding its *unitary irreducible representations* (unirreps), which constitute the possible *quantizations*. Although the actual procedure for finding unirreps might not be what really matters from the physical point of view we proceed along a well-defined algorithm, the group approach to quantization or GAQ for brief, to obtain them for any Lie group.

All the ingredients of GAQ are canonical structures defined on Lie groups and the very basic ones consist in the *two* mutually *commuting* copies of the Lie algebra $\tilde{\mathcal{G}}$ of a group \tilde{G} of *strict symmetry* (of a given physical system), that is, the set of left- and right-invariant vector fields:

$$\mathcal{X}^L(\tilde{G}) \approx \tilde{\mathcal{G}} \approx \mathcal{X}^R(\tilde{G})$$

in such a way that one copy, let us say $\mathcal{X}^R(\tilde{G})$, plays the role of *pre-Quantum Operators* acting (by usual derivation) on complex (wave) functions on \tilde{G} , whereas the other, $\mathcal{X}^L(\tilde{G})$, is used to *reduce* the representation in a manner *compatible* with the action of the operators, thus providing the *true quantization*.

In fact, from the group law $g'' = g' * g$ of any group \tilde{G} , we can read two different left- and right-actions:

$$g'' = g' * g \equiv L_{g'}g, \quad g'' = g' * g \equiv R_g g'. \quad (6)$$

Both actions commute and so do their respective generators \tilde{X}_a^R and \tilde{X}_b^L , i.e. $[\tilde{X}_a^L, \tilde{X}_b^R] = 0 \ \forall a, b$.

Another manifestation of the commutation between left and right translations corresponds to the invariance of the left-invariant canonical 1-forms, $\{\theta^{La}\}$ (dual to $\{\tilde{X}_b^L\}$, i.e. $\theta^{La}(\tilde{X}_b^L) = \delta_b^a$) with respect to the right-invariant vector fields, that is: $L_{\tilde{X}_a^R}\theta^{Lb} = 0$ and the other way around ($L \leftrightarrow R$). In particular, we dispose of a natural *invariant volume* ω on the group manifold since we have:

$$L_{\tilde{X}_a^R}(\theta^{Lb} \wedge \theta^{Lc} \wedge \theta^{Ld} \dots) \equiv L_{\tilde{X}_a^R}\omega = 0. \quad (7)$$

We should then be able to recover all *physical ingredients* of quantum systems out of *algebraic structures*. In particular, the Poincaré-Cartan form Θ_{PC} and the phase space itself $M \equiv (x^i, p_j)$ should be regained from a group of *strict symmetry* \tilde{G} . In fact, in the special case of a Lie group which bears a central extension with structure group $U(1)$ parameterized by $\zeta \in C$ such

that $|\zeta|^2 = 1$, as we are in fact considering, the group manifold \tilde{G} itself can be endowed with the structure of a *principal bundle* with an *invariant connection*, thus generalizing the notion of *quantum manifold*.

More precisely, the $U(1)$ -component of the left-invariant canonical form (dual to the *vertical* generator \tilde{X}_ζ^L , i.e. $\theta^{L(\zeta)}(\tilde{X}_\zeta^L) = 1$) will be named *quantization form* $\Theta \equiv \theta^{L(\zeta)}$ and generalizes the Poincaré-Cartan form Θ_{PC} of Classical Mechanics. The quantization form remains *strictly invariant* under the group \tilde{G} in the sense that

$$L_{\tilde{X}_a^R} \Theta = 0 \quad \forall a$$

whereas Θ_{PC} is, in general, only *semi-invariant*, that is to say, it is invariant except for a total differential.

It should be stressed that the construction of a true quantum manifold in the sense of Geometric Quantization [2, 3] can be achieved by taking in the pair $\{\tilde{G}, \Theta\}$ the quotient by the action of the subgroup generated by those left-invariant vector fields in the kernel of Θ and $d\Theta$, that which is called in mathematical terms *characteristic module* of the 1-form Θ ,

$$\mathcal{C}_\Theta \equiv \{\tilde{X}^L / i_{\tilde{X}^L} d\Theta = 0 = i_{\tilde{X}^L} \Theta\}.$$

A further quotient by structure subgroup $U(1)$ provides the *classical solution Manifold* M or classical *phase space*. Even more, the vector fields in \mathcal{C}_Θ constitute the (generalized) *classical equations of motion*.

On the other hand, the right-invariant vector fields are used to provide classical functions on the phase space. In fact, the functions

$$F_a \equiv i_{\tilde{X}_a^R} \Theta \tag{8}$$

are stable under the action of the left-invariant vector fields in the characteristic module of Θ , the equations of motion,

$$L_{\tilde{X}^L} F_a = 0 \quad \forall \tilde{X}^L \in \mathcal{C}_\Theta$$

and then constitute the *Noether invariants*.

As a consequence of the central extension structure in \tilde{G} the Noether invariants (and the corresponding group parameters) are classified in basic (symplectic or dynamical) and non-basic (non-symplectic or kinematic) depending on whether or not the corresponding generators produce the central generator by commutation with some other. Basic parameters (Noether invariants) are paired (and independent). Non-basic Noether invariants (like energy or angular momenta) can be written in terms of the basic ones (positions and momenta).

As far as the quantum theory is concerned, the above-mentioned quotient by the classical equations of motion is really not needed. We consider the space of complex functions Ψ on the whole group \tilde{G} and restrict them to only $U(1)$ -functions, that is, those which are homogeneous of degree 1 on the argument $\zeta \equiv e^{i\phi} \in U(1)$. Wave functions thus satisfy the $U(1)$ -function condition

$$\tilde{X}_\phi^L \Psi = i\Psi. \tag{9}$$

On these functions the right-invariant vector fields act as *pre-quantum operators* by ordinary derivation. They are, in fact, Hermitian operators with respect to the scalar product with measure given by the invariant volume ω defined above (7). However, this action is not a proper

quantization of the Poisson algebra of the Noether invariants (associated with the symplectic structure given by $d\Theta$) since there is a set of non-trivial operators commuting with this representation. In fact, all the left-invariant vector fields do commute with the right-invariant ones, i.e. the pre-quantum operators, and, therefore, the representation is not irreducible. According to Schur's Lemma those operators must be trivialized. To this end we define a polarization subalgebra as follows:

A polarization \mathcal{P} is a maximal left subalgebra containing the characteristic subalgebra \mathcal{G}_Θ and excluding the central generator.

The role of a polarization is that of *reducing* the representation which then constitutes a true *quantization*. To this end we impose on wave functions the polarization condition:

$$\tilde{X}_b^L \Psi = 0 \quad \forall \tilde{X}_b^L \in \mathcal{P}$$

In finite terms the polarization condition is expressed by the invariance of the wave functions under the finite action of the Polarization Subgroup G_P acting from the right, that is:

$$\Psi(g'g_P) = \Psi(g') \quad \forall g_P \in G_P. \quad (10)$$

To be intuitive, a polarization is made of half the left-invariant vector fields associated with basic (independent) variables of the solution manifold in addition to those associated with non-symplectic parameters as time or rotational angles. We should remark that the classification above-mentioned of the Noether invariants in basic and non-basic also applies to the quantum operators so that the latter ones are written in terms of the formers.

As an additional comment regarding polarization conditions, it must be stressed that when expressed as quantum equations, they contain, in particular, the evolution equation properly, that is, the Schrodinger(-like) equation. In this respect these polarizations (and the GAQ method itself) depart from those in Geometric Quantization, which are imposed only after having taken the quotient by the classical evolution explicitly, that which means having solved the classical equations. Another respect on which GAQ departs from GQ is in that the entire enveloping algebra (both left and right ones) can be used to construct higher-order Polarizations and higher-order operators.

The integration volume ω can be restricted to the Hilbert space of polarized wave functions \mathcal{H} by means of a canonical procedure a bit technical for the scope of the present report. We refer the reader to Ref.[17].

Before ending this section let us mention that the existence of a polarization containing the entire characteristic subalgebra (usually referred to as full polarization) is not guaranteed in general and we then can resort to the left enveloping algebra to complete the polarization in the same way that any operator in the right enveloping algebra can be properly realized as a quantum operator (see Ref. [18]). Higher-order polarizations are used by strict necessity, when no full polarization can be found (in this case the system is *anomalous* in the standard physical sense [19]), or simply by pure convenience of realizing the quantization in a particular "representation" adapted to given variables.

4 Quantum Mechanics (examples with a finite number of degrees of freedom)

In this section some examples of *symmetry groups* \leftrightarrow *physical systems* correspondence involving a finite number of dynamical variables are reported in the simplest manner, showing the way GAQ can be used in practice. Formal developments and subtleties are left for a further reading of the references.

4.1 The Free Galilean Particle

We shall adopt the notation $B \equiv t$, $A \equiv x$, $V \equiv v$ ($p \equiv mv$) for the parameters in the group law to reinforce the fact that all physical variables do emerge naturally from the group manifold itself and the dimension will be kept to $1 + 1$ to reduce the expressions to the minimum.

Reading the group law (4) in the new variables and deriving the double primed variables with respect to every non-primed and primed one at the identity we get the explicit expressions of the left- and right-vector fields, respectively:

$$\begin{aligned} \tilde{X}_t^L &= \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \frac{1}{2}mv^2 \frac{\partial}{\partial \phi} & \tilde{X}_t^R &= \frac{\partial}{\partial t} \\ \tilde{X}_x^L &= \frac{\partial}{\partial x} & \tilde{X}_x^R &= \frac{\partial}{\partial x} + mv \frac{\partial}{\partial \phi} \\ \tilde{X}_v^L &= \frac{\partial}{\partial v} + mx \frac{\partial}{\partial \phi} & \tilde{X}_v^R &= \frac{\partial}{\partial v} + t \frac{\partial}{\partial x} + mtv \frac{\partial}{\partial \phi} \\ \tilde{X}_\phi^L &= \frac{\partial}{\partial \phi} & \tilde{X}_\phi^R &= \frac{\partial}{\partial \phi} \end{aligned} \quad (11)$$

By duality on the left generators, selecting the $U(1)$ component, or by using the direct formula

$$\Theta \equiv \theta^{L\phi} = \frac{\partial \phi''}{\partial g} \Big|_{g'=g^{-1}, g}, \quad (12)$$

one can compute the quantization form (the actual expression of the Poincaré-Cartan part is defined up to a total differential depending of the particular co-cycle used in the group law, which is defined in turns up to a co-boundary; see Ref.[14, 1]):

$$\Theta \equiv \theta^{L\phi} = -mxdv - \frac{1}{2}mv^2 dt + d\phi$$

From the commutation relations (the left ones change the structure constants by a global sign)

$$\left[\tilde{X}_t^R, \tilde{X}_x^R \right] = 0, \quad \left[\tilde{X}_t^R, \tilde{X}_v^R \right] = \tilde{X}_x^R, \quad \left[\tilde{X}_x^R, \tilde{X}_v^R \right] = -m\tilde{X}_\phi^R \quad (13)$$

one rapidly identifies x, v as canonically conjugated (symplectic) variables and t as a non-symplectic parameter. In fact, the left generator \tilde{X}_t^L generates the characteristic subalgebra \mathcal{G}_Θ and constitutes the classical equations of motion (generalized, since there is an extra equation for the central parameter).

The quantum wave functions are complex functions on \tilde{G} , $\Psi = \Psi(\zeta, x, v, t)$, restricted by the $U(1)$ -function condition (9), as well as the polarization conditions

$$\tilde{X}_a^L \Psi = 0 \quad (a = t, x \text{ maximal set})$$

We then obtain:

$$\begin{aligned}\tilde{X}_\phi^L \Psi &= i\Psi \Rightarrow \Psi = \zeta \Phi(t, x, v) \\ \tilde{X}_x^L \Psi &= 0 \Rightarrow \Phi \neq \Phi(x), \Phi = \varphi(t, v) \\ \tilde{X}_t^L \Psi &= 0 \Rightarrow \frac{\partial \varphi}{\partial t} + \frac{i}{2} m v^2 \varphi = 0 \Rightarrow i \frac{\partial \varphi}{\partial t} = \frac{p^2}{2m} \varphi,\end{aligned}$$

i.e. the Schrödinger equation in momentum space.

On the (reduced) wave functions the right-invariant vector fields act reproducing the standard quantum operators in momentum space “representation”:

$$\tilde{X}_x^R \varphi = i m v \varphi, \quad \tilde{X}_v^R \varphi = \frac{\partial}{\partial v} \varphi, \quad \tilde{X}_t^R \varphi = -i \frac{p^2}{2m} \varphi, \quad (14)$$

the operator $\hat{E} \equiv i \tilde{X}_t^R$ being a function of the basic one $\hat{p} \equiv i \tilde{X}_x^R$. Had we considered the motion in $3 + 1$ dimensions, we would have found also new operators in the characteristic subalgebra associated with rotations acquiring the usual expressions in terms of the basic operators \hat{v} and $\hat{x} \equiv i \tilde{X}_v^R$.

4.2 Revisited Minimal Coupling Principle

In this subsection we attempt to describe group-theoretically the motion of a particle subject to an external field. Even though we do not intend to account for the field degrees of freedom, the transformation properties of its “zero-modes” can be encoded into part of a symmetry group. The general mechanism under which a free particle starts suffering an interaction parallels the well-known Minimal Coupling Principle, which is now revisited from our group-theoretical approach. We shall be concerned here with the classical domain only.

Let \tilde{G} be a quantization group generated by $\{\tilde{X}_A\}$, $A = 1, \dots, n$ and $\{\tilde{X}_a\}$, $a = 1, \dots, m < n$ an invariant subalgebra:

$$[\tilde{X}_A, \tilde{X}_a] = C_{Aa}^b \tilde{X}_b$$

If we make “local” the subgroup generated by $\{\tilde{X}_a\}$, that is to say, if the corresponding group variables are allowed to depend arbitrarily on the space-time parameters, we get an infinite-dimensional Lie algebra:

$$\{f^a \otimes \tilde{X}_a, \tilde{X}_A\}$$

with the following new commutators:

$$\begin{aligned}[\tilde{X}_A, f^a \otimes \tilde{X}_a] &= f^a \otimes [\tilde{X}_A, \tilde{X}_a] + L_{\tilde{X}_A} f^a \otimes \tilde{X}_a \\ &= f^a \otimes C_{Aa}^b \tilde{X}_b + L_{\tilde{X}_A} f^a \otimes \tilde{X}_a\end{aligned} \quad (15)$$

Now we just attempt to “quantize” this new (local) group $\tilde{G}(\vec{x}, t)$.

4.2.1 Particle in an Electromagnetic Field.

We start from the $U(1)$ -extended Galilei group, \tilde{G} , and make the rigid group $\zeta = e^{i\phi} \in U(1)$ into “local”, i.e. we allow the parameter to depend on the space-time variables, $\phi = \phi(\vec{x}, t)$. The idea is to keep the invariance of the generalized Poincaré-Cartan form $\Theta = p_i dx^i - \frac{\vec{p}^2}{2m} dt + d\phi$ under the locally extended Galilei group.

According to the *Revisited Minimal Coupling Principle* [20] we only have to compute the 1-form Θ associated with the Galilei group extended by the infinite dimensional group $U(1)(\vec{x}, t)$. But, in order to parameterize properly the quantization group let us formally write

$$\phi(\vec{x}, t) = \phi(0, 0) + \phi_\mu(\vec{x}, t)x^\mu \equiv \phi + A_\mu(\vec{x}, t)x^\mu \quad (16)$$

and compute the group law:

$$\begin{aligned} t'' &= t' + t \\ \vec{x}'' &= \vec{x}' + R'\vec{x} + \vec{v}'t \\ \vec{v}'' &= \vec{v}' + R'\vec{v} \\ A_{\vec{x}}'' &= A_{\vec{x}}' + R'A_{\vec{x}} \\ A_t'' &= A_t' + A_t + \vec{v}' \cdot R'A_{\vec{x}} \\ \phi'' &= \phi' + \phi + \mathbf{m}[\vec{x}' \cdot R'\vec{v} + t(\vec{v}' \cdot R'\vec{v} + \frac{1}{2}v'^2)] + \mathbf{q}[\vec{x}' \cdot R'A_{\vec{x}} + t\vec{v}' \cdot R'A_{\vec{x}} + tA_t'] \end{aligned}$$

where two different co-cycles characterized by m and q , that is, the mass and the electric charge, have been introduced.

From now on we shall disregard the rotation subgroup although the vector character of the variables will be maintained. Also, and since we do not intend to describe quantum aspects, the expression of the left-invariant generators will be omitted (see Ref. [20]) and only the Lie algebra commutators are written:

$$\begin{aligned} \left[\tilde{X}_t^L, \tilde{X}_{\vec{x}}^L \right] &= 0 & \left[\tilde{X}_{x^i}^L, \tilde{X}_{A_{x^j}}^L \right] &= \mathbf{q}\delta_{ij}\tilde{X}_\phi^L & \left[\tilde{X}_t^L, \tilde{X}_{\vec{v}}^L \right] &= -\tilde{X}_{\vec{x}}^L \\ \left[\tilde{X}_{\vec{x}}^L, \tilde{X}_{A_t}^L \right] &= 0 & \left[\tilde{X}_t^L, \tilde{X}_{A_x}^L \right] &= 0 & \left[\tilde{X}_v^L, \tilde{X}_{A_x}^L \right] &= \tilde{X}_{A_t}^L \\ \left[\tilde{X}_t^L, \tilde{X}_{A_t}^L \right] &= -\mathbf{q}\tilde{X}_\phi^L & \left[\tilde{X}_v^L, \tilde{X}_{A_t}^L \right] &= 0 & \left[\tilde{X}_{x^i}^L, \tilde{X}_{v^j}^L \right] &= \mathbf{m}\delta_{ij}\tilde{X}_\phi^L \end{aligned} \quad (17)$$

By duality from the explicit expression of the left-invariant generators, derived in turn from the group law, or directly from the composition law corresponding to the $U(1)$ parameter, through the formula (12), we obtain the quantization form

$$\Theta = -m\vec{x} \cdot d\vec{v} - q\vec{x} \cdot d\vec{A} - \left(\frac{1}{2}m\vec{v}^2 + qA_t\right)dt + d\phi$$

whose characteristic module contains the generator of the time evolution: X such that $i_X\Theta = i_X d\Theta = 0$, that is,

$$X = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} - \frac{q}{m} \left[\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right) v^j + \frac{\partial A_0}{\partial x^i} + \frac{\partial A_i}{\partial t} \right] \frac{\partial}{\partial v_i}$$

which implies the following explicit equations of motion:

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad m \frac{d\vec{v}}{dt} = q[\vec{v} \wedge (\vec{\nabla} \wedge \vec{A}) - \vec{\nabla} A_0 - \frac{\partial \vec{A}}{\partial t}]. \quad (18)$$

Making the standard change of variables

$$\vec{B} \equiv \vec{\nabla} \wedge \vec{A}, \quad \vec{E} \equiv -\vec{\nabla} A_0 - \frac{\partial \vec{A}}{\partial t} \quad (19)$$

we finally arrive at the ordinary equation of a particle suffering the Lorentz force:

$$m \frac{d\vec{v}}{dt} = q[\vec{E} + \vec{v} \wedge \vec{B}]$$

As a last general comment, let us remark once again the physical relevance of central extensions. It might seem paradoxical the fact that a non-trivial vector potential (in the sense that it is not the gradient of a function) can be derived some how from the function $\phi(\vec{x}, t)$, but it is the central extension mechanism what insures that A_μ can be something different from the gradient of a scalar function. In other words, the generator X_{A_μ} in (17), for $q = 0$, necessarily generates trivial (gauge) changes in A_μ .

4.2.2 Particle in a gravitational field.

Let us pass very briefly through this finite-dimensional example where the computations are made in dimension $1 + 1$ although the vector notation is restored at the end. We start from the $U(1)$ -extended Poincaré group and make “local” the translation subgroup, the Lie algebra of which can be written as

$$\begin{aligned} \begin{cases} [\tilde{X}_t^R, \tilde{X}_x^R] = 0 \\ [\tilde{X}_t^R, \tilde{X}_v^R] = \tilde{X}_x^R \\ [\tilde{X}_x^R, \tilde{X}_v^R] = -\frac{1}{c^2} \tilde{X}_t^R - m \tilde{X}_\phi^R \end{cases} & \quad \text{or} \quad \begin{cases} [P_0, P] = 0 \\ [P_0, K] = P \\ [P_0, K] = -\frac{1}{c^2} P_0 - m X_\phi, \end{cases} \end{aligned}$$

and repeat the process of “localizing” translations in a way analogous to that followed for the $U(1)$ subgroup in the electromagnetic case. We write, as before,

$$f^\mu \otimes P_\mu = (f^\mu(0) + f^{\mu\sigma}(x)x_\sigma) \otimes P_\mu$$

and rename the functions $f^{\mu\nu}$ as $h^{\mu\sigma}$, which will prove to be the non-Minkowskian part of a non-trivial metric, that is: $h^{\mu\nu} \equiv g^{\mu\nu} - \eta^{\mu\nu}$

The Lie algebra must be explicitly written according to the general formula (15) and the rigid algebra (extended Poincaré). We show in boldface the terms that survive after an Inönü-Wigner contraction with respect to the subgroup generated by \tilde{X}_t (the non-relativistic limit):

$$\begin{aligned} \begin{cases} [\tilde{X}_v, \tilde{X}_x] = -\tilde{X}_t + \mathbf{mc}\tilde{\mathbf{X}}_\phi \\ [\tilde{X}_x, \tilde{X}_{h_{0x}}] = -\tilde{X}_t - \mathbf{g}\tilde{\mathbf{X}}_\phi \\ [\tilde{X}_{h_{00}}, \tilde{X}_{h_{0x}}] = \tilde{X}_v \\ [\tilde{X}_t, \tilde{X}_{h_{00}}] = \tilde{X}_t + \mathbf{g}\tilde{\mathbf{X}}_\phi \end{cases} & \quad \begin{cases} [\tilde{X}_t, \tilde{X}_{h_{0x}}] = \tilde{\mathbf{X}}_x \\ [X_v, \tilde{X}_{h_{00}}] = -\tilde{X}_{h_{0x}} \\ [X_v, \tilde{X}_{h_{xx}}] = \tilde{X}_{h_{0x}} \\ [X_v, \tilde{X}_{h_{0x}}] = -\tilde{X}_{h_{00}} + \tilde{X}_{h_{xx}} \end{cases} & \quad \begin{cases} [\tilde{X}_v, \tilde{X}_t] = -\tilde{\mathbf{X}}_x \\ [\tilde{X}_x, \tilde{X}_{h_{xx}}] = -\tilde{X}_x \\ [\tilde{X}_{h_{0x}}, \tilde{X}_{h_{xx}}] = \tilde{X}_v \end{cases} \end{aligned}$$

It should be remarked that we have naively written a *gravitational coupling constant* g in places that parallel those of the electric charge q in the Lie algebra that accounts for the

electromagnetic interaction; that is to say, on the right hand side of the commutators $[\tilde{X}_t, \tilde{X}_{h_{00}}]$ and $[\tilde{X}_x, \tilde{X}_{h_{0x}}]$, but the Jacobi identity requires the equality $g = mc$, which may be properly identified with an algebraic version of the **Equivalence Principle**. Note that both q and $g = mc$ are true central charges (in the sense that they parameterize non-trivial central extensions) in the non-relativistic limit. To be precise, q also parameterizes non-trivial central extension in the Poincaré group.

Now, the group law must be computed order by order, although it is enough to keep the expansion up to the 3th order for illustrating the dynamics. We remit the readers to Ref. [20] for a detailed computation and here only the final equation of motion are showed.

Geodesic Force: We introduce for simplicity the vector notation: $h^{0i} \equiv \vec{h}$. In terms of these variables the equations of motion, for low gravity and low velocity, are:

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad m \frac{d\vec{v}}{dt} = -m \left[\vec{v} \wedge (\vec{\nabla} \wedge \vec{h}) - \vec{\nabla} h^{00} - \frac{\partial \vec{h}}{\partial t} \right] + \frac{m}{4} \vec{\nabla} (\vec{h} \cdot \vec{h}) \quad (20)$$

They reproduce the standard geodesic motion, up to the limits mentioned, and in a form that emulate the electromagnetic motion (18) for electromagnetic-like vector potential $\mathcal{A} = (h^{00} - \frac{1}{4}\vec{h} \cdot \vec{h}, \vec{h})$ according to that which is named “gravitoelectromagnetic” description in the literature (see, for instance, Ref. [21, 22, 23, 24]).

4.3 Particle moving on a group manifold: case of the $SU(2)$ group

In this last finite-dimensional example let us adopt a slightly different point of view, that is, we shall start from the classical Lagrangian and try to close a Poisson subalgebra containing $\langle H, q, p \rangle$. However, except for simple examples such a subalgebra is infinite. To keep ourselves in finite dimensions (that is, with a finite number of degrees of freedom) we may alternatively resort to an auxiliary, different finite-dimensional Poisson subalgebra (closing a group G) such that, in its enveloping algebra, the original functions $\langle H, q, p \rangle$ can be found, and therefore quantized. In fact, in the GAQ scheme, not only the generators of the original group G can be quantized, but also the entire universal enveloping algebra. This procedure has been explicitly achieved in dealing with the quantum dynamics of a particle in a (modified) Pöschl-Teller potential[25], where the “first-order” (auxiliary) group G used was $SL(2, R)$.

We start by parametrizing rotations with a vector $\vec{\varepsilon}$ in the rotation-axis direction and with modulus

$$|\vec{\varepsilon}| = 2 \sin \frac{\varphi}{2}$$

$$R(\vec{\varepsilon})_j^i = (1 - \frac{\vec{\varepsilon}^2}{2}) \delta_j^i - \sqrt{1 - \frac{\vec{\varepsilon}^2}{4}} \eta_{jk}^i \varepsilon^k + \frac{1}{2} \varepsilon^i \varepsilon_j$$

In these coordinates the canonical left-invariant 1-forms read:

$$\theta_j^{L(i)} = \left[\sqrt{1 - \frac{\vec{\varepsilon}^2}{4}} \delta_j^i + \frac{\varepsilon^i \varepsilon_j}{4 \sqrt{1 - \frac{\vec{\varepsilon}^2}{4}}} + \frac{1}{2} \eta_{jm}^i \varepsilon^m \right]$$

and in terms of these the *particle- σ -Model Lagrangian* acquires the following expression:

$$\mathcal{L} = \frac{1}{2} \delta_{ij} \theta_m^{L(i)} \theta_n^{L(j)} \dot{\varepsilon}^m \dot{\varepsilon}^n = \frac{1}{2} \left[\delta_{ij} + \frac{\varepsilon_i \varepsilon_j}{4(1 - \frac{\vec{\varepsilon}^2}{4})} \right] \dot{\varepsilon}^i \dot{\varepsilon}^j \equiv \frac{1}{2} g_{ij} \dot{\varepsilon}^i \dot{\varepsilon}^j$$

Proceeding much in the same way followed in the previous section, we compute the canonical momenta:

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}^i} = g_{ij} \dot{\varepsilon}^j$$

and the Hamiltonian:

$$\mathcal{H} = \pi_i \dot{\varepsilon}^i - \mathcal{L} = \frac{1}{2} g^{-1ij} \pi_i \pi_j$$

We assume the canonical bracket between the basic functions ε^i and π_j :

$$\begin{aligned} \{\varepsilon^i, \pi_j\} &= \delta_j^i \quad \text{added with} \\ \{\mathcal{H}, \varepsilon^i\} &= -g^{-1ij} \pi_j \\ \{\mathcal{H}, \pi_i\} &= \frac{1}{2} (\vec{\varepsilon} \cdot \vec{\pi}) \pi_i, \end{aligned}$$

so that $\langle \mathcal{H}, \varepsilon^i, \pi_j, 1 \rangle$ do not close a finite-dimensional Lie algebra.

However, we may define the following set of new “coordinates”, “momenta” and even, “energy” and “angular momenta”:

$$\langle p^i \equiv 2g^{-1ij} \pi_j, k^j \equiv \sqrt{2\mathcal{H}} \varepsilon^j, E \equiv 2\sqrt{2\mathcal{H}}, J^k \equiv \eta_{mn}^k \varepsilon^m \pi^n \rangle.$$

They close the Lie algebra of $SO(3, 2)$ i.e. an Anti-de Sitter algebra. That is, the basic brackets:

$$\begin{aligned} \{E, p^i\} &= k^i \\ \{E, k^j\} &= -p^j \\ \{k^i, p_j\} &= \delta_j^i E, \end{aligned}$$

along with the induced ones:

$$\begin{aligned} \{k^i, k^j\} &= -\eta^{ij}_{\cdot k} J^k & \{p^i, p^j\} &= -\eta^{ij}_{\cdot k} J^k \\ \{J^i, J^j\} &= \eta^{ij}_{\cdot k} J^k & \{J^i, k^j\} &= \eta^{ij}_{\cdot k} k^k \\ \{J^i, p^j\} &= \eta^{ij}_{\cdot k} p^k & \{E, J^j\} &= 0, \end{aligned}$$

close a finite-dimensional Lie algebra to which we may apply the GAQ. (Note the minus sign in the first line, which states that the involved group is $SO(3, 2)$ and not $SO(4, 1)$).

We then quantize the Anti-de Sitter group so that the original operators $\hat{\mathcal{H}}, \hat{\pi}_i, \hat{\varepsilon}^j$ can be found in its *enveloping algebra* through the expression:

$$\langle \hat{\mathcal{H}} \equiv \frac{1}{8} \hat{E}^2, \hat{\pi}^i \equiv \frac{1}{4} (\hat{g}^{ij} \hat{p}_j + \hat{p}_j \hat{g}^{ij}), \varepsilon^i \equiv \frac{1}{2\sqrt{2}} \left(\hat{\mathcal{H}}^{-1/2} \hat{k}^i + \hat{k}^i \hat{\mathcal{H}}^{-1/2} \right), \hat{1} \rangle$$

In so doing we parameterize a central extension of the Anti-de Sitter group by (abstract) variables $\{a^0, \vec{a}, \vec{v}, \vec{\varepsilon}, \zeta\}$, which mimic those for the Poincaré group. In the same way we hope that the right-invariant generators associated with those parameters reproduce corresponding functions $\{E, p_i, k^j, J^k, 1\}$ as Noether invariants satisfying the Poisson brackets above.

At this point it should be stressed that the parameters $\vec{\varepsilon}$ and the corresponding quantum operators (essentially the right generators \hat{X}_ε^R) are associated with ordinary rotations on Anti-de Sitter space-time, whereas the $\vec{\varepsilon}$ parameters correspond to “translations” on the $SU(2)$ manifold.

We shall not give here the explicit group law for the group variables nor the explicit expression for the left-invariant vector fields on the extended $SO(3, 2)$ group (which can be found in Refs. [26, 27]), limiting ourselves to write the explicit expression for the (higher-order) polarization condition and the wave functions as solutions of it. That is, from the polarization condition $\mathcal{P}\psi = 0$, with

$$\mathcal{P} = \langle \tilde{X}_{a^0}^{LHO} \equiv (\tilde{X}_{a^0}^L)^2 - c^2(\tilde{X}_{\vec{a}}^L)^2 - \frac{2imc^2}{\hbar}\tilde{X}_{a^0}^L, \tilde{X}_{\vec{\nu}}^L, \tilde{X}_{\vec{\epsilon}}^L \rangle$$

we arrive at a wave function depending only on a^0 , \vec{a} and satisfying a Klein-Gordon-like equation with $SO(3, 2)$ D'Alembertian operator given by

$$\begin{aligned} \square = & \frac{1}{16q_a^2} \{ [16 - (a^0)^2 \frac{\omega^2}{c^2} (8 + \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2))] \frac{\partial^2}{\partial a^{02}} - a^0 \frac{\omega^2}{c^2} [40 + 7 \frac{\omega^2}{c^2} (r_a^2 \\ & - (a^0)^2)] \frac{\partial}{\partial a^0} - [16 + r_a^2 \frac{\omega^2}{c^2} (8 + \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2))] \frac{\partial^2}{\partial r_a^2} - \frac{1}{r_a} [32 + \frac{\omega^2}{c^2} (40 \\ & + 7 \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2))] \frac{\partial}{\partial r_a} 2a^0 r_a \frac{\omega^2}{c^2} (8 + \frac{\omega^2}{c^2} (r_a^2 - (a^0)^2)) \frac{\partial^2}{\partial a^0 \partial r_a} \} + \frac{\vec{L}^2}{q_a^2 r_a^2}, \end{aligned}$$

where

$$\vec{L}^2 = -\frac{1}{\sin\theta_a} \frac{\partial}{\partial\theta_a} (\sin\theta_a \frac{\partial}{\partial\theta_a}) - \frac{1}{\sin^2\theta_a} \frac{\partial^2}{\partial\varphi_a^2}$$

is the square of the standard orbital angular momentum operator (save for a factor \hbar) and

$$r_a = \sqrt{\vec{a} \cdot \vec{a}}, \quad q_a = \sqrt{1 + \frac{\omega^2}{4c^2} (\vec{a}^2 - (a^0)^2)}. \quad (21)$$

The *wave functions* are

$$\phi(\vec{a}, a^0) = e^{-2ic\lambda_{nl} \arcsin(\frac{\omega q_a a^0}{\sqrt{4c^2 + \omega^2 q_a^2 r_a^2})} Y_m^l(\theta_a, \varphi_a) (1 + \frac{\omega^2}{c^2} q_a^2 r_a^2)^{-\frac{\lambda_{nl}}{2}} (q_a r_a)^l \phi_l^{\lambda_{nl}}(q_a r_a)$$

where

$$\begin{aligned} \lambda_{nl} & \equiv \frac{E}{\hbar\omega} \\ E & \equiv (\frac{3}{2} + 2n + l + \frac{1}{2} \sqrt{9 + 4 \frac{m^2 c^2}{\hbar^2 \omega^2} - 48\xi}) \hbar\omega \\ \phi_l^{\lambda_{nl}} & = {}_2F_1(-n, n + l + \frac{3}{2} - \lambda_{nl}, l + \frac{3}{2}; -\frac{\omega^2}{c^2} q_a^2 r_a^2) \end{aligned}$$

and ξ is a free parameter related to the “zero-point energy” [28]. On this representation the operators corresponding to the original functions ε^i , π_j and the energy \mathcal{H} can be realized.

5 Quantum Field Theory (examples with an infinite number of degrees of freedom)

Typical infinite-dimensional systems in Physics appear as mappings from a space-time manifold M into a (not necessarily Abelian) target group G

$$\varphi : M \rightarrow G, \quad x \mapsto \varphi(x). \quad (22)$$

If a is an invertible, differentiable transformation of M , i.e. a is an element in $\text{Diff}(M)$, or a subgroup of it, the following semi-direct $(\text{Diff}(M) \otimes_s G(M))$ group law holds:

$$a'' = a' \circ a, \quad \varphi''(x) = \varphi'(a(x)) * \varphi(x), \quad (23)$$

where \circ is the composition group law in $\text{Diff}(M)$ (composition of mappings), $*$ denotes the composition group law in the target group G and $a(x)$ stands for the action of $\text{Diff}(M)$ on M . When the group G is not a (complex) vector space \mathbb{C}^n , the group of mappings is usually called gauge, local or current group $G(M)$. Specially well-known are the unitary gauge groups on Minkowski space-time and the loop groups which correspond to the case in which M is the circle S^1 . However, the actual physical fields correspond to the elements in the centrally extended group $\widehat{\text{Diff}(M)} \otimes_s G(M)$. In the case of $M = S^1$ the group (23) has a specially rich structure (that is Virasoro \otimes_s Kac-Moody) [29] with many applications in conformal field theory in 1 + 1 dimensions [30]. As a general comment, the ability in parametrizing the infinite-dimensional group (23) will play a preponderant role in the corresponding physical description.

5.1 The Klein-Gordon Field

As a very simple example of the general scheme above-mentioned let us consider the case in which M is the Minkowski space-time with co-ordinates $(x^0 \equiv ct, \vec{x})$, $\text{Diff}(M)$ is restricted to its Poincaré subgroup (or even just the space-time translations subgroup, for the sake of simplicity), parameterized by $(a^0 \equiv cb, \vec{a})$, and G is simply a complex vector space, let us say \mathbb{C} , parameterized by φ .

There is a natural parametrization of the group above associated with a factorization of M as $\Sigma \times R$, that is, the product of a Cauchy surface Σ and the time real line. In fact, we can use $\langle b, \vec{a}; \varphi(\vec{x}), \dot{\varphi}(\vec{x}) \rangle$. In these variables, however, whereas the space translations \vec{a} act on $\varphi(\vec{x})$ by just moving the arguments as $\vec{x} + \vec{a}$: $\varphi(\vec{x} + \vec{a}) = \exp(i\vec{a} \cdot \partial_{\vec{x}})\varphi(\vec{x})$, making b an action on $(\varphi(\vec{x}), \dot{\varphi}(\vec{x}))$ requires the knowledge of the equation of motion, though not necessarily their solutions. For the Klein-Gordon field the time evolution equations are

$$\ddot{\varphi}(\vec{x}) = (\vec{\nabla}^2 - m^2)\varphi(\vec{x})$$

and the time action $\varphi'(b(\vec{x}))$ reads:

$$\varphi'(b(\vec{x})) \equiv e^{ibc\partial_0}\varphi'(\vec{x}) = \cos\left[bc\sqrt{m^2 - \vec{\nabla}^2}\right]\varphi'(\vec{x}) + i\frac{\sin\left[bc\sqrt{m^2 - \vec{\nabla}^2}\right]}{\sqrt{m^2 - \vec{\nabla}^2}}\dot{\varphi}'(\vec{x})$$

This way, all canonical operations on groups can be easily performed. For instance, the right-invariant vector fields are:

$$\begin{aligned} X_b^R &= \frac{\partial}{\partial b} \\ X_{\varphi(\vec{x})}^R &= \cos\left(b\sqrt{m^2 - \vec{\nabla}^2}\right)\frac{\delta}{\delta\varphi(\vec{x})} - \sqrt{m^2 - \vec{\nabla}^2}\sin\left(b\sqrt{m^2 - \vec{\nabla}^2}\right)\frac{\delta}{\delta\dot{\varphi}(\vec{x})} \\ X_{\dot{\varphi}(\vec{x})}^R &= \cos\left(b\sqrt{m^2 - \vec{\nabla}^2}\right)\frac{\delta}{\delta\dot{\varphi}(\vec{x})} + \frac{1}{\sqrt{m^2 - \vec{\nabla}^2}}\sin\left(b\sqrt{m^2 - \vec{\nabla}^2}\right)\frac{\delta}{\delta\varphi(\vec{x})}, \end{aligned}$$

and their commutation relations:

$$\begin{aligned} [X_b^R, X_{\varphi(\vec{x})}^R] &= -(m^2 - \vec{\nabla}^2) X_{\dot{\varphi}(\vec{x})}^R \\ [X_b^R, X_{\dot{\varphi}(\vec{x})}^R] &= X_{\varphi(\vec{x})}^R \\ [X_{\varphi(\vec{x})}^R, X_{\dot{\varphi}(\vec{x}')}^R] &= 0 \quad (\delta(\vec{x} - \vec{x}') X_{\phi}^R \text{ when centrally extended}). \end{aligned}$$

Notice that the actual solutions of the equations of motion of a more general system are not required since the corresponding Lie algebra can be exponentiated (at least) order by order giving rise to the finite action of b on both $\varphi(\vec{x})$ and $\dot{\varphi}(\vec{x})$.

As mentioned above, what really matters for the physical description is the corresponding centrally extended group. In order to motivate such extension we shall proceed in a way analogous to that followed in the case of Mechanics. Let us go then temporarily to the standard Lagrangian formalism for classical fields. The real Klein-Gordon field of mass m is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) \quad (24)$$

which is well-known to realize the Poincaré symmetries (see, for instance Ref. [31]). However, the Noether invariants associated with space-time symmetries are not relevant in studying the solution manifold \mathcal{M} in the sense that they are not the basic, independent functions parametrizing the phase space. In fact, quantities such as the energy-momentum tensor or the generalized (rotations and Lorentz) angular momenta are written in terms of the Fourier coefficients $a(k)$, $a^*(k)$, where the four vector k_μ runs on the Lorentz orbit $k^\mu k_\mu = m^2$. Here we are primarily interested in characterizing those Fourier coefficients as Noether invariants of certain generators leaving semi-invariant the Lagrangian (24). To this end we consider the following vector fields on the complete (including the field derivatives) configuration space for the Klein-Gordon Field $(x^\nu, \varphi, \varphi_\mu)$:

$$\bar{X}_{a^*(k)} \equiv i e^{ikx} \frac{\partial}{\partial \varphi} - k_\nu i e^{ikx} \frac{\partial}{\partial \varphi_\nu} \quad (25)$$

Computing the Lie derivative of the Lagrangian with respect to this vector (note that the second components of this vector are simply the derivatives of the components on φ) we obtain:

$$L_{\bar{X}_{a^*(k)}} \mathcal{L} = \partial_\mu \beta^\mu, \quad \beta^\mu \equiv -k^\mu e^{ikx} \varphi, \quad (26)$$

where explicit use of the mass-shell condition for k has been made. The Noether theorem establishes that the current

$$J_{a^*(k)}^\mu = X_{a^*(k)}^\varphi \pi^\mu - \beta^\mu$$

(where $X_{a^*(k)}^\varphi$ is the φ -component of the generator, i.e. the infinitesimal variation $\delta\varphi$ and $\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \varphi_\mu}$ is the field covariant momentum) is conserved: $\partial_\mu J^\mu = 0$. The Noether charge reads:

$$Q_{X_{a^*(k)}} = \int d^3x J^0 = i \int d^3x e^{ikx} (\dot{\varphi} - ik^0 \varphi) \quad (27)$$

which turns out to be just the Fourier coefficient $a(k)$. In the same way we obtain the charge $a^*(k)$ and so a coordinate system for the solution manifold \mathcal{M} made of Noether invariants.

Analogously, an equivalent configuration-space parametrization can be considered. In fact, the vector fields:

$$\begin{aligned} X_{\pi(\vec{y})} &\equiv i \int \frac{d^3k}{2k^0} \left[e^{i\vec{k}\cdot\vec{y}} e^{ikx} \frac{\partial}{\partial\varphi} - h.c \right] \\ X_{\varphi(\vec{y})} &\equiv - \int \frac{d^3k}{2k^0} k^0 \left[e^{i\vec{k}\cdot\vec{y}} e^{ikx} \frac{\partial}{\partial\varphi} + h.c \right] \end{aligned} \quad (28)$$

have as Noether invariants the values of φ and $\pi = \dot{\varphi}$ on each one of the points of the Cauchy surface Σ . In terms of these “configuration-space” variables the symplectic form on \mathcal{M} adopts the aspect and properties of that of Classical Mechanics:

$$w_{K-G} = d\vartheta_{K-G} = d \left(\int d^3x \pi(\vec{x}) \delta\varphi(\vec{x}) \right),$$

the symplectic potential ϑ_{K-G} , or Liouville form, being *semi-invariant under the basic symmetries* (28).

To end up with the (semi-)invariance properties of the Klein-Gordon field it should be mentioned that the symmetries of this Lagrangian can be given the aspect of some sort of “residual gauge” symmetry, even for $m \neq 0$. In fact, for any real function f on M satisfying the Klein-Gordon equation, the vector field

$$X^f = f \frac{\partial}{\partial\varphi} + f_\mu \frac{\partial}{\partial\varphi_\mu}$$

leaves (24) semi-invariant.

Once the necessity of a central extension of the semi-invariance group (23) has been stated and motivated, we write the quantization group for the Klein-Gordon field in covariant form [32] as follows:

$$\begin{aligned} a'' &= a' + \Lambda' a \\ \Lambda'' &= \Lambda' \Lambda \\ \varphi''(x) &= \varphi'(\Lambda x + a) + \varphi(x) \\ \varphi_\mu''(x) &= \varphi_\mu'(\Lambda x + a) + \varphi_\mu(x) \\ \zeta'' &= \zeta' \zeta \exp \left\{ \frac{i}{2} \int_\Sigma d\sigma^\mu [\varphi'(\Lambda x + a) \varphi_\mu(x) - \varphi_\mu'(\Lambda x + a) \varphi(x)] \right\}, \end{aligned}$$

where the Poincaré subgroup is parameterized by (a, Λ) , $d\sigma^\mu = n^\mu(x) d\sigma$ and x is, in principle, supposed to live on the Cauchy hypersurface Σ , with unit normal vector $n^\mu(x)$, although the fields φ and φ_μ can be defined on the entire Minkowski space-time if they satisfy the equations of motion. In this sense, the exponential is well-defined, even though the fields would live on the whole space-time, because the integrand is a conserved current. Further details can be found in [32] and references therein.

5.2 Sigma Model-type systems

A less trivial example of infinite dimensions arises when the target group, in which the fields are valued, is considered to be non-Abelian. Then a physical system associated with such fields is the so-called Non-Linear Sigma Model (NLSM).

For the sake of simplicity, let us consider the case of $G = SU(2)$. Denoting by τ_a , $a = 1, 2, 3$, the Lie algebra basis matrices, with commutation relations $[\tau_a, \tau_b] = \eta_{ab}^c \tau_c$, and $U = \exp(i\varphi^b \tau_b)$ an element of $SU(2)$ parameterized by $\varphi \equiv (\varphi^1, \varphi^2, \varphi^3)$, the left-invariant canonical 1-form [for the mappings $\varphi^b(x) \in G$ in (22)] is $\theta_\mu^L = U^{-1} \partial_\mu U \equiv \theta_\mu^{L(b)} \tau_b$. In terms of $\theta_\mu^L \equiv (\theta_\mu^{L(1)}, \theta_\mu^{L(2)}, \theta_\mu^{L(3)})$, the Lagrangian of the $SU(2)$ NLSM can be written as:

$$\mathcal{L}_\sigma = \frac{1}{2} \text{Tr}_G(\theta_\mu^L \theta^{L\mu}) \quad (29)$$

This Lagrangian is invariant under the left action (6) of $SU(2)$ and, eventually, it is right-invariant as well (i.e. it is chiral).

In facing the group-quantization of these systems we find a serious obstruction: It is not possible (without drastically distorting the leading $SU(2)$ symmetry) to find a semi-invariance –like in (26)– helping us to identify the quantization group, even in the solution manifold. This can be illustrated by means of the action of the would-be semi-invariance generator in the Abelian case, $\frac{\partial}{\partial \theta_\mu^L}$, on the symplectic potential (covariantly written in a spatial hypersurface Σ):

$$\begin{aligned} \vartheta_\sigma &\equiv \int_\Sigma d\sigma_\mu \pi_a^\mu d\varphi^a = \int_\Sigma d\sigma_\mu \frac{\partial \mathcal{L}_\sigma}{\partial \varphi_\mu^a} d\varphi^a = \int_\Sigma d\sigma_\mu \theta_a^{L\mu} \theta^{La}, \\ L_{\frac{\partial}{\partial \theta_\mu^L}} \vartheta_\sigma &= \int_\Sigma d\sigma_\mu \theta^L \end{aligned}$$

But $\theta^L \equiv U^{-1} dU$ is no longer a total differential when dealing with non-Abelian Lie groups; indeed, it verifies $\partial_\mu \theta_\nu^L - \partial_\nu \theta_\mu^L = [\theta_\mu^L, \theta_\nu^L] \neq 0$. So, we are not able to give the complete dynamical symmetry of the system on the spot.

Although the procedure followed above in the finite-dimensional case might be formally applied, the normal ordering ambiguities eventually appearing are more involved in field theory, giving rise to problems quite analogous to the non-renormalizability arising in the usual canonical quantization approach. However, this symmetry obstruction does disappear if we restrict to NLSM on just co-adjoint-like orbits of the corresponding group, governed by a *partial trace* Lagrangian.

5.2.1 Partial trace.

Let us replace the target manifold for the fields in (29) (i.e. $G \equiv SU(2)$) by a coset G/G_λ , $G_\lambda = U(1)$ being the isotropy subgroup of a given Lie algebra element $\lambda = \lambda^a \tau_a$ under the adjoint action $\lambda \rightarrow g\lambda g^{-1}$ of $SU(2)$. To be precise λ should have been defined as an element of the dual of the Lie algebra, which is equivalent to the Lie algebra since $SU(2)$ is semisimple. Now, the (total-trace) NLSM Lagrangian (29) is replaced by the partial-trace one:

$$\mathcal{L}_{G/G_\lambda} = \frac{1}{2} \text{Tr}_{G/G_\lambda}(\theta_\mu^L \theta^{L\mu}) \equiv \frac{1}{2} \text{Tr}_G([\lambda, \theta_\mu^L][\lambda, \theta^{L\mu}]). \quad (30)$$

It can be realized that, defining

$$\Lambda(x) \equiv U(x) \lambda U(x)^{-1} \equiv \Lambda^a(x) \tau_a, \quad U(x) \in G,$$

we have an alternative way of writing (30) as

$$\mathcal{L}_{G/G_\lambda} = \frac{1}{2} \text{Tr}_G(\Lambda_\mu \Lambda^\mu) = \frac{1}{2} K_{ab} \Lambda_\mu^a \Lambda^{b\mu}, \quad (31)$$

where K_{ab} is the Killing metric. Note that this Lagrangian is singular due to the existence of constraints like, for example, $\text{Tr}_G(\Lambda(x)^2) = \text{Tr}_G(\lambda^2) \equiv \text{constant}$. We shall not deal with constraints at this stage. They will be naturally addressed inside our quantization procedure below.

The Lagrangian (31) generalizes to field theory that of a point particle constrained to move on a sphere. In [33] it was shown that the Euclidean Group constitutes the addressing symmetry in the case of \mathbb{S}^2 , achieving the expected quantum theory.

5.2.2 Local Euclidean Group.

The field analogue to the symmetry of a particle moving on the sphere \mathbb{S}^2 proves to be a local version of the Euclidean group and its unitary and irreducible representations are intended to account for the quantization of the partial-trace NLSM associated with the group $G = SU(2)$ (the extension to any semi-simple Lie group G is essentially straightforward). This group will be parameterized by the local $SU(2)$ parameters $\varphi^a(x)$ and the (co-)adjoint parameters $\theta_\mu^L{}^a(x)$ (the superscript L will be omitted in the sequel). Then, the vector fields $X_{\varphi^a(x)}$, $a = 1, 2, 3$, generate local internal rotations and the vector fields $X_{\theta_\mu^L{}^a(x)}$, $a = 1, 2, 3$, $\mu = 0, 1, 2, 3$, generate the (co-)tangent subgroup. In terms of these variables the group law is:

$$\begin{aligned} U''(x) &= U'(x)U(x) \\ \theta_\mu''(x) &= U'(x) \theta_\mu(x) U'^\dagger(x) + \theta'_\mu(x) \\ \zeta'' &= \zeta' \zeta \exp \left\{ i \int_\Sigma d\sigma^\nu \text{Tr} \left[\lambda \left(U'(x) \theta_\nu(x) U'^\dagger(x) - \theta_\nu(x) \right) \right] \right\}, \end{aligned}$$

where $U(x) \equiv U(\varphi(x))$. Here, all fields are assumed to be defined on the Cauchy surface Σ , so that, the time translation can not be directly implemented, in contrast with the Klein-Gordon case. However, we shall construct an explicit Hamiltonian operator to account for the time evolution on the quantum states (see below).

We can immediately compute the corresponding right-invariant vector fields:

$$\begin{aligned} \tilde{X}_{\varphi^a(x)}^R &= X_{\varphi^a(x)}^{R(G)} - \eta_{ab}{}^c \theta_\mu^b(x) \frac{\delta}{\delta \theta_\mu^c(x)} + \eta_{ab}{}^c \theta_\mu^b(x) \lambda_c n^\mu \Xi \\ \tilde{X}_{\theta_\mu^a(x)}^R &= \frac{\delta}{\delta \theta_\mu^a(x)} \\ \tilde{X}_\zeta^R &= \text{Re}(i\zeta \frac{\partial}{\partial \zeta}) \equiv \Xi, \end{aligned} \quad (32)$$

closing the Lie algebra of the local Euclidean Group:

$$\begin{aligned} [\tilde{X}_{\varphi^a(x)}^R, \tilde{X}_{\varphi^b(y)}^R] &= -\eta_{ab}{}^c \delta(x-y) \tilde{X}_{\varphi^c(x)}^R \\ [\tilde{X}_{\varphi^a(x)}^R, \tilde{X}_{\theta_\mu^b(y)}^R] &= -\eta_{ab}{}^c \delta(x-y) \left(\tilde{X}_{\theta_\mu^c(x)}^R - \lambda_c n^\mu \Xi \right) \\ [\tilde{X}_{\theta_\mu^a(x)}^R, \tilde{X}_{\theta_\nu^b(y)}^R] &= 0. \end{aligned} \quad (33)$$

We can also compute the left-invariant vector fields:

$$\begin{aligned}\tilde{X}_{\varphi(x)}^L &= X_{\varphi(x)}^{L(G)} \\ \tilde{X}_{\theta_\mu(x)}^L &= U \frac{\delta}{\delta \theta_\mu(x)} U^\dagger - (U^\dagger \lambda U - \lambda) n^\mu \Xi \\ \tilde{X}_\zeta^L &= \text{Re}(i\zeta \frac{\partial}{\partial \zeta}) \equiv \Xi,\end{aligned}\tag{34}$$

closing the same Lie algebra except for opposite structure constants. Directly from the group law or by duality on (34) the left-invariant 1-form in the ζ -direction, Θ_σ , can be computed

$$\Theta_\sigma^{G/G_\lambda} = - \int_\Sigma d\sigma^\nu \text{Tr}((\Lambda - \lambda)\delta\theta_\nu) + \frac{d\zeta}{i\zeta}.\tag{35}$$

Note that only the time-like component of those θ_μ in the coadjoint orbit defined by λ will contribute to the solution manifold. In fact, the characteristic module is generated by:

$$\mathcal{G}_{\Theta_\sigma^{G/G_\lambda}} = \langle \lambda^a \tilde{X}_{\varphi^a(x)}^L, \lambda^b n_\mu \tilde{X}_{\theta_\mu^b(x)}^L, \tilde{X}_{\theta_\nu^c(x)}^L - n^\nu \tilde{X}_{\theta_\rho^c(x)}^L n^\rho \rangle.$$

According to the general formalism, Sec. 3, the Noether invariants are the following:

$$\begin{aligned}I_\varphi &= [\Lambda n_\mu, \theta^\mu] \equiv \mathbb{L} \\ I_{\theta^\mu} &= (\Lambda - \lambda) n_\mu,\end{aligned}\tag{36}$$

although only \mathbb{L} and $\mathbb{S} \equiv n^\mu I_{\theta^\mu}$ are independent, the basic ones, and parameterize the solution manifold. In terms of these Noether invariants, the quantization 1-form can be expressed as

$$\Theta_\sigma^{G/G_\lambda} = \int_\Sigma d\sigma^\nu \text{Tr}(\mathbb{S} \delta[\mathbb{L}, I_{\theta^\nu}]) + \frac{d\zeta}{i\zeta}.\tag{37}$$

The basic Poisson brackets read

$$\begin{aligned}\{\mathbb{L}_a(\vec{x}), \mathbb{L}_b(\vec{y})\} &= -\eta_{ab}{}^c \mathbb{L}_c(\vec{x}) \delta(\vec{x} - \vec{y}), \\ \{\mathbb{L}_a(\vec{x}), \mathbb{S}_b(\vec{y})\} &= -\eta_{ab}{}^c \mathbb{S}_c(\vec{x}) \delta(\vec{x} - \vec{y}) + \eta_{ab}^c \lambda_c \delta(\vec{x} - \vec{y}).\end{aligned}\tag{38}$$

Even though we have not included the Poincaré subgroup, the time evolution (we shall choose $\Sigma = \mathbb{R}^3$ in the time direction, i.e., $d\sigma_\mu \rightarrow d^3x$) can be realized on the solution manifold by giving a Hamiltonian function of the basic Noether invariants. The Hamiltonian,

$$\mathbb{H} = \frac{1}{2} \int d^3x \{ \mathbb{L}^2 + (\vec{\nabla} \mathbb{S})^2 \},\tag{39}$$

reproduces the classical equations of motion in terms of the Poisson brackets above.

In order to achieve the quantization of the present system, we must find a polarization sub-algebra. It is given by the characteristic module together with half of the conjugated pairs:

$$\mathcal{P} = \langle \lambda^a \tilde{X}_{\varphi^a(x)}^L, n_\mu \tilde{X}_{\theta_\mu^b(x)}^L, \tilde{X}_{\theta_\nu^c(x)}^L - n^\nu \tilde{X}_{\theta_\rho^c(x)}^L n^\rho \rangle.$$

An irreducible representation of the group is given by the action of the right-invariant vector fields of the group on the complex functions valued over the group manifold, provided that these functions are polarized and satisfy the U(1)-function condition:

$$\mathcal{P}\Psi = 0, \quad \Xi\Psi = i\Psi.$$

It can be easily checked that such functions (now true *wave functions*) are of the form:

$$\psi = \zeta e^{-i \int_{\Sigma} d\sigma^{\mu} \text{Tr}(\lambda(U^{\dagger}\theta_{\mu}U - \theta_{\mu}))} \Phi((\Lambda - \lambda)) \quad (40)$$

where Φ is an arbitrary function of its argument.

The action of the right-invariant vector fields preserve the space of polarized wave functions, due to the commutativity of the left and right actions as already stated, so that it is possible to define an action of them on the arbitrary factor Φ in the wave functions. It is not difficult to demonstrate that on this space of functions the basic quantum operators acquire the following expression:

$$\begin{aligned} \hat{S}_a \Phi &\equiv i\zeta^{-1} e^{i \int_{\Sigma} d\sigma^{\mu} \text{Tr}(\lambda(U^{\dagger}\theta_{\mu}U - \theta_{\mu}))} n_{\nu} \tilde{X}_{\theta_{\nu}^a}^R \psi = (\mathbb{S}_a - \lambda_a) \Phi \\ \hat{L}_a \Phi &\equiv -i\zeta^{-1} e^{i \int_{\Sigma} d\sigma^{\mu} \text{Tr}(\lambda(U^{\dagger}\theta_{\mu}U - \theta_{\mu}))} \tilde{X}_{\varphi^a}^R \psi = \eta_{ab}^{c} \mathbb{S}^b \frac{\delta}{\delta \mathbb{S}^c} \Phi, \end{aligned} \quad (41)$$

where the operator $n_{\mu} \tilde{X}_{\theta_{\mu}^a(x)}^R$ can be redefined with the addition of λ to fix the vacuum expectation value to zero.

On the quantum representation space we can construct the Hamiltonian operator

$$\hat{H}\Phi = \frac{1}{2} \int d^3x \{ \hat{L}^2 + (\vec{\nabla} \hat{S})^2 \} \Phi, \quad (42)$$

that represents the classical Hamiltonian \mathbb{H} without ordering ambiguity, due to its quadratic expression in terms of the basic operators. It must be emphasized that this operator preserves the Hilbert space of quantum states.

Thus, it becomes evident at this point that the domain of the wave functions has been naturally selected without imposing any constraint condition as such. Our quantization program chooses as the basic observables the $\tilde{X}_{\varphi(x)}^R$ operators, playing the role of “generators of translations” in the internal parameter space, and the conjugated ones $n_{\mu}(x) \tilde{X}_{\theta_{\mu}(x)}^R$, playing that of “field” operators. Note that had we resorted to canonical quantization we should have postulated basic commutators of the generic form $[\varphi(x), \frac{\partial}{\partial \varphi(y)}] = \delta(x - y)$, in deep contrast with (33).

Let us remark that even though we have discarded the Poincaré symmetry as a subgroup of our quantization group \tilde{G} , the extended local Euclidean group, and in particular the time translation, we have provided a quantum Hamiltonian preserving the Hilbert space of the quantum representation. This means that we can proceed with any computation involving the time evolution without disturbing the physical system as a whole. Namely, we can construct a perturbative theory in the “Heisenberg picture” [34], that is to say, evolving the wave functions, originally defined on the Cauchy hypersurface, with the exponential of the total Hamiltonian. The explicit perturbation series can be achieved according to the Magnus expansion [35] which guarantees unitarity to each order. Also, if the actual Hamiltonian can be decomposed into two pieces each one preserving the original Hilbert space, one of them considered as a free Hamiltonian \hat{H}_0 , the other as an interaction Hamiltonian, \hat{H}_{int} , it is then possible to achieve the

perturbation theory in two steps. In that case, in the first step we arrive at some sort of “free” theory to which we apply the perturbation addressed by \hat{H}_{int} , in a way that realizes the more standard Dyson expansion in the “interaction picture”. In Ref. [36] we analyzed this scheme and discussed the differences with ordinary perturbation approach, in particular the fact that we start from a Hamiltonian \hat{H}_0 whose classical theory possesses the same topology as that of the total Hamiltonian, a relevant ingredient to achieve a unitary, renormalized theory.

This group-quantization of non-linear sigma models is being further developed in relation with relevant physical applications in alternatives to the Higgs-Kibble mechanism of mass generation [37] (see also [38, 39]).

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